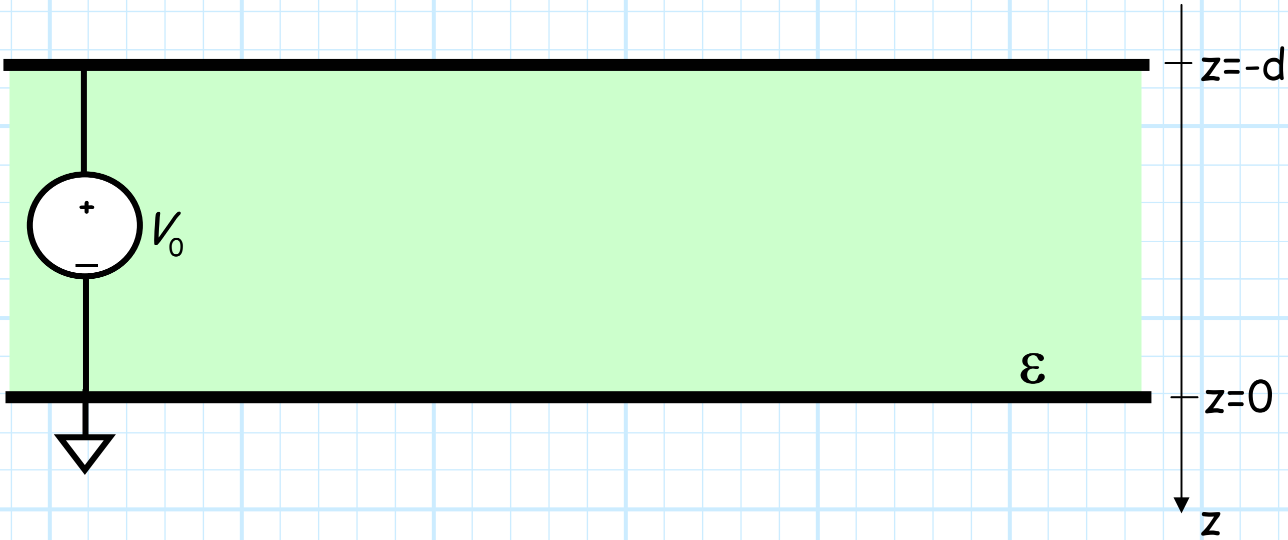
Example: Dielectric Filled Parallel Plates

Consider two infinite, parallel **conducting** plates, spaced a distance *d* apart. The region between the plates is filled with a dielectric ε. Say a voltage V_0 is placed across these plates.



Q: *What electric potential field $V(\bar{r})$, electric field $\mathbf{E}(\bar{r})$ and charge density $\rho_s(\bar{r})$ is produced by this situation?*

A: We must solve a **boundary value problem** ! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Q: *Yikes! Where do we even start ?*

A: We might start with the electric potential field $V(\bar{\mathrm{r}})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's equation**:

$$\nabla^2 V(\bar{\mathrm{r}}) = \frac{-\rho_v(\bar{\mathrm{r}})}{\varepsilon}$$

b) It must also satisfy the **boundary conditions**:

$$V(z=-d) = V_0 \qquad\qquad V(z=0) = 0$$

Consider first the dielectric region ($-d < z < 0$). Since the region is a dielectric, there is **no** free charge, and:

$$\rho_v(\bar{\mathrm{r}}) = 0$$

Therefore, Poisson's equation reduces to **Laplace's equation**:

$$\nabla^2 V(\bar{\mathrm{r}}) = 0$$

This problem is greatly simplified, as it is evident that the solution $V(\bar{\mathrm{r}})$ is independent of coordinates x and y. In other words, the electric potential field will be a function of coordinate z **only**:

$$V(\bar{\mathrm{r}}) = V(z)$$

This make the problem **much** easier! Laplace's equation becomes:

$$\nabla^2 V(\bar{r}) = 0$$
$$\nabla^2 V(z) = 0$$
$$\frac{\partial^2 V(z)}{\partial z^2} = 0$$

Integrating **both** sides of Laplace's equation, we get:

$$\int \frac{\partial^2 V(z)}{\partial z^2}\ dz = \int 0\ dz$$
$$\frac{\partial V(z)}{\partial z} = C_1$$

And integrating **again** we find:

$$\int \frac{\partial V(z)}{\partial z}\ dz = \int C_1\ dz$$
$$V(z) = C_1 z + C_2$$

We find that the equation $V(z) = C_1 z + C_2$ **will** satisfy Laplace's equation (try it!). We must now apply the **boundary conditions** to determine the value of constants C_1 and C_2.

We know that the value of the electrostatic potential at every point on the top (z=-d) plate is $V(-d)= V_0$, while the electric potential on the bottom plate (z=0) is zero ($V(0) = 0$). Therefore:

$$V(z=-d) = -C_1 d + C_2 = V_0$$

$$V(z=0) = C_1(0) + C_2 = 0$$

Two equations and **two** unknowns (C_1 and C_2)!

Solving for C_1 and C_2 we get:

$$C_2 = 0 \quad \text{and} \quad C_1 = -\frac{V_0}{d}$$

and therefore, the **electric potential** field within the dielectric is found to be:

$$V(\bar{r}) = \frac{-V_0 z}{d} \qquad (-d \le z \le 0)$$

Before we proceed, let's do a **sanity check**!

In other words, let's evaluate our answer at $z = 0$ and $z = -d$, to make **sure** our result is correct:

$$V(z=-d) = \frac{-V_0(-d)}{d} = V_0 \quad \checkmark$$

and

$$V(z=0) = \frac{-V_0(0)}{d} = 0 \quad \checkmark$$

Now, we can find the **electric field** within the dielectric by taking the **gradient** of our result:

$$\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{\mathbf{r}}) = \frac{V_0}{d}\hat{a}_z \quad (-d \le z \le 0)$$

And thus we can easily determine the **electric flux density** by multiplying by the dielectric of the material:

$$\mathbf{D}(\bar{\mathbf{r}}) = \varepsilon\mathbf{E}(\bar{\mathbf{r}}) = \frac{\varepsilon V_0}{d}\hat{a}_z \quad (-d \le z \le 0)$$

Finally, we need to determine the **charge density** that actually created these fields!

Q: *Charge density !?! I thought that we already determined that the charge density $\rho_v(\bar{\mathbf{r}})$ is equal to* ***zero****?*

A: We know that the free charge density **within the dielectric** is zero—but there must be charge **somewhere**, otherwise there would be no fields!

Recall that we found that **at** a conductor/dielectric **interface**, the **surface charge density** on the conductor is related to the **electric flux density** in the dielectric as:

$$D_n = \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}}) = \rho_s(\bar{\mathbf{r}})$$

First, we find that the electric flux density on the **bottom** surface of the **top** conductor (i.e., at $z = -d$) is:

$$\mathbf{D}(\bar{\mathbf{r}})\Big|_{z=-d} = \frac{\varepsilon V_0}{d}\hat{a}_z \Bigg|_{z=-d} = \frac{\varepsilon V_0}{d}\hat{a}_z$$

For **every** point on **bottom** surface of the **top** conductor, we find that the unit vector **normal** to the conductor is:

$$\hat{a}_n = \hat{a}_z$$

Therefore, we find that the **surface charge density** on the bottom surface of the top conductor is:

$$\begin{aligned}\rho_{s+}(\bar{\mathbf{r}}) &= \hat{a}_n \cdot \mathbf{D}(\bar{\mathbf{r}})\Big|_{z=-d} \\ &= \hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} \\ &= \frac{\varepsilon V_0}{d} \qquad (z = -d)\end{aligned}$$

Likewise, we find the unit vector **normal** to the **top** surface of the **bottom** conductor is (do you see why):

$$\hat{a}_n = -\hat{a}_z$$

Therefore, evaluating the **electric flux density** on the top surface of the bottom conductor (i.e., $z = 0$), we find:

$$\begin{aligned}\rho_{s-}(\bar{r}) &= \hat{a}_n \cdot \mathbf{D}(\bar{r})\Big|_{z=0} \\ &= -\hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} \\ &= \frac{-\varepsilon V_0}{d} \qquad (z = 0)\end{aligned}$$

We should **note** several things about these solutions:

1) $\nabla \times \mathbf{E}(\bar{r}) = 0$

2) $\nabla \cdot \mathbf{D}(\bar{r}) = 0$ and $\nabla^2 V(\bar{r}) = 0$

3) $\mathbf{D}(\bar{r})$ and $\mathbf{E}(\bar{r})$ are **normal** to the surface of the conductor (i.e., their **tangential** components are equal to **zero**).

4) The **electric field** is precisely the **same** as that given by using superposition and eq. 4.20 in section 4-5!

I.E.:

$$\mathbf{E}(\bar{r}) = \frac{\rho_{s+}}{2\varepsilon}\hat{a}_z - \frac{\rho_{s-}}{2\varepsilon}\hat{a}_z = \frac{V_0}{d}\hat{a}_z \quad (-d < z < 0)$$

In other words, the **fields** $\mathbf{E}(\bar{r})$, $\mathbf{D}(\bar{r})$, and $V(\bar{r})$ are attributable to **charge densities** $\rho_{s+}(\bar{r})$ and $\rho_{s-}(\bar{r})$.